APPENDIX G

The Pre-Inflationary Bridge: Scaling, Duration, and Semiclassical Applicability

G.1 Purpose and Scope

This appendix collects the quantitative bridge relations between ρ_{crit} and ρ_{inf} for the radiation-like equation of state wbridge = 1/3 established in §9.2.2. It derives the exact density and energy scaling, the bridge duration in the adopted effective LQC dynamics, and the classicalization window. It also compares the externally identified first-EFT density with the characteristic energy scale of the representative Starobinsky Stage III realization.

G.2 Exact Bridge Scaling

Local conservation with $w = 1/3$ gives

$$\rho a^4 = \text{const}, \qquad Ea = \text{const}$$

$$\alpha = \frac{E_{\mathrm{EFT}}}{E_{\mathrm{b}}} = \left(\frac{\rho_{\mathrm{inf}}}{\rho_{\mathrm{crit}}}\right)^{1/4} \simeq 8.52 \times 10^{-4}$$

$$N_{\mathrm{pre}} = -\ln\alpha \simeq 7.07$$

Thus α and N_{pre} are two representations—multiplicative and logarithmic—of the same bridge density hierarchy, not independent confirmations.

G.3 CMB-Normalized Starobinsky Scale

For the representative Starobinsky realization,

$$M \simeq M_{\mathrm{Pl,red}} \frac{\sqrt{24\pi^2 A_{\mathrm{s}}}}{N_\star}$$

With $A_{\mathrm{s}} \approx 2.1 \times 10^{-9}$ and $N_* = 50$–60,

$$M \simeq (2.86\text{–}3.43) \times 10^{13}\ \mathrm{GeV}$$

$$V_\star^{1/4} \simeq (7.8\text{–}8.5) \times 10^{15}\ \mathrm{GeV}$$

corresponding to

$$\rho_{\mathrm{St},\star} \simeq (7.6\text{–}10.9) \times 10^{100}\ \mathrm{J\,m^{-3}}$$

The displayed range results from varying the adopted N_* interval (50–60) at fixed CMB-normalized scalar amplitude A_{s}. It is a model-sensitivity range, not an observational standard-deviation error or a statistical confidence interval.

This characteristic Stage III scale overlaps the Grand-Unification neighborhood used for the external identification of ρ_{inf}. The numerical proximity provides the intended consistency check between the first-EFT scale and the representative inflationary realization.

G.4 Physical Duration of the Bridge

In the adopted effective LQC dynamics the radiation-like bridge obeys the exact solution

$$\rho(t) = \frac{\rho_{\mathrm{crit}}}{1 + At^2}, \qquad A = \frac{32\pi \mathrm{G}\rho_{\mathrm{crit}}}{3\mathrm{c}^2}$$

With $t = 0$ at the bounce and $\rho(t_{\mathrm{inf}}) = \rho_{\mathrm{inf}}$,

$$t_{\mathrm{inf}} \simeq 2.0 \times 10^{-38}\ \mathrm{s} \simeq 3.72 \times 10^{5}\ t_{\mathrm{Pl}}$$

The physical duration is therefore fixed by the endpoint densities and the adopted effective bounce dynamics, rather than by a divergent classical extrapolation through $t = 0$.

G.5 Classicalization Window and Semiclassical Applicability

The LQC correction factor $\rho/\rho_{\mathrm{crit}}$ falls rapidly after the bounce. The representative window $N \approx 0.8$–1.6 corresponds to $t \approx 1.3$–$6.6\ t_{\mathrm{Pl}}$ and reduces the correction from a few-percent level to well below one percent. Beyond this early interval, the effective bridge is accurately described by the radiation-like semiclassical scaling while the density continues toward ρ_{inf}.

Irreversibility remains a prior admissibility condition; the classicalization window identifies when its semiclassical description becomes operationally adequate.

G.6 Suppression of Pre-Inflationary Radiative Relics

G.6.1 Radiative Production at the End of the Bridge

Dynamical particle production by the time-dependent bridge background is not universal: conformally invariant massless fields in a flat FLRW background are not excited by the expansion (Parker 1969; Birrell and Davies 1982). The following analysis tracks the dilution of a relic component present at the onset of Stage III.

G.6.2 Dilution During Conditional Stage III

For a relic radiation component present at the onset of the accelerated Stage III,

$$\frac{\rho_{\mathrm{relic,end}}}{\rho_{\mathrm{relic,onset}}} = \exp(-4N_{\mathrm{III}})$$

where N_{III} is the total number of accelerated e-folds between Stage III onset and the end of inflation.

The observable quantity $N_* \approx 50$–60 instead measures pivot exit to the end of inflation. If Stage III is already active when the pivot exits, $N_{\mathrm{III}} \geq N_*$, so even the pivot-to-end interval alone gives a suppression between approximately e^{-200} and e^{-240}. For the representative $N_* = 58$ the factor is $\mathrm{e}^{-232} \approx 1.8 \times 10^{-101}$.

G.6.3 Present-Day Frequency Benchmark

For a freely propagating relativistic mode with characteristic production energy E_{prod} near the first-EFT endpoint, the frequency redshifts as $f \propto a^{-1}$. If the mode is already present at the onset of accelerated Stage III, then after N_{III} accelerated e-folds,

$$f_{\mathrm{end}} \simeq \frac{E_{\mathrm{prod}}}{h}\exp(-N_{\mathrm{III}})$$

Let

$$N_{\mathrm{reh}} = \ln\left(\frac{a_{\mathrm{reh}}}{a_{\mathrm{end}}}\right)$$

denote the expansion between the end of inflation and the reference end of reheating. Subsequent expansion to the present is included in the frequency relation below. The quantities a_{end}, a_{reh}, and a_0 are scale factors at the end of Stage III, at the chosen end-of-reheating reference, and today; only their dimensionless ratios enter. E_{prod} is the mode production energy and h is Planck's constant, so E_{prod}/h has units of frequency. N_{III} and N_{reh} are dimensionless expansion e-fold counts; N_* instead counts e-folds from the pivot-scale horizon exit to the end of inflation.

$$f_0 \simeq \frac{E_{\mathrm{prod}}}{h}\exp(-N_{\mathrm{III}})\exp(-N_{\mathrm{reh}})\frac{a_{\mathrm{reh}}}{a_0}$$

For the historical prompt-reheating benchmark used in this work, take $E_{\mathrm{prod}} \approx 8.2 \times 10^{15}$ GeV, $N_{\mathrm{III}} = 58$, $N_{\mathrm{reh}} = 0$, and $a_0/a_{\mathrm{reh}} \approx 10^{28}$. Then

$$f_0 \simeq 1.3 \times 10^{-14}\ \mathrm{Hz}$$

Equivalently, the benchmark scaling may be written

$$f_0 \simeq 1.3 \times 10^{-14}\ \mathrm{Hz}\left(\frac{E_{\mathrm{prod}}}{8.2 \times 10^{15}\ \mathrm{GeV}}\right)\exp[-(N_{\mathrm{III}} - 58)]\exp(-N_{\mathrm{reh}})\left[\frac{10^{28}}{a_0/a_{\mathrm{reh}}}\right]$$

This frequency is a conditional propagation benchmark for a relativistic mode produced near the first-EFT endpoint and subsequently redshifted through the specified expansion history. The normalization uses a representative accelerated duration N_{III}=58, prompt reheating, and the stated post-reheating expansion. It does not determine the peak or amplitude of a stochastic gravitational-wave background, which would additionally require a specified production mechanism and source spectrum. The total duration N_{III} is distinct from N_*; different accelerated durations and reheating histories shift the benchmark.

G.6.4 Conclusion

The accelerated Stage III, when realized with sufficient duration, suppresses pre-existing bridge relics by an enormous factor. Conformally invariant massless fields constitute the limiting case in which the expansion does not generate such quanta in the first place.

G.7 Structural Summary

The radiation-like bridge fixes α, N_{pre}, t_{inf}, and the classicalization window from ρ_{crit}, ρ_{inf}, $w = 1/3$, and the adopted effective LQC dynamics. The Starobinsky comparison introduces a separate Stage III scale, $\rho_{\mathrm{St},*}$, whose CMB normalization lies in the same broad high-energy neighborhood as the external first-EFT identification. The bridge is complete before that model-dependent Stage III realization is specified.